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Approximate Solution of Initial Value Problems by Means of Ninth Degree Spline

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Abstract:

In this paper, a nonlinear initial value problem is solved numerically by means of ninth degree spline function. The solution of initial value problems approximated as a linear interpolation of ninth spline functions. In this method, the basis spline functions are redefined into a new approximation set of ninth degree spline functions which interpolate the number of select derivatives. To test the efficiency of the method, two numerical examples of initial value problems are solved by the proposed method.

Key words and phrases: Spline function, convergence analysis, bounded errors, nonlinear IVPs.

I. Introduction:

Interpolation spline functions have many applications in various fields of science and engineering such as biological models, industrial mathematics, economics, electrostatics, fluid dynamics and so on (See [2, 5, 8, 12]). It is well known that it is extremely difficult to analytically solve nonlinear ordinary differential equations. Indeed, few of these equations can be solved explicitly. So it is required to devise an efficient approximation scheme for solving these equations.

The aim of this paper is to construct a spline function based on a lacunary interpolation by spline function of ninth degree to develop numerical methods for obtaining smooth approximations for the solution of the linear and nonlinear initial value problems. Recently, a new method based on various degree of spline function is used to develop numerical solution for initial value problems, see for example [6, 7, 10, 11]. As an extension of Saeed,et al. [12], we propose in this paper ninth degree spline function method for solvingthe following nonlinear initial value problems numerically:

$$\frac{d^3u}{dt^3} = f(t, u, u', u'')$$

$$u(t_0) = u_0,\ u'(t_0) = u'_0 \ \text{ and } \ u''(t_0) = u''_0 \qquad (1)$$

II. Description of the Method:

Consider the function $f(x)$ be a solution of any third order and second order linear or nonlinear differential equations, and the ninth degree spline is denoted by $p_\Delta(x)$ and defined on $I = [a,b]$ as:.

$$p_0(x) = y_0 + hy_0' + \frac{h^2}{2}y_0'' + \frac{h^3}{6}y_0'' + \frac{h^4}{24}y_0^{(4)} + h^5 a_{0,5} + h^6 a_{0,6} + \frac{x^7}{5040}y_0^{(7)} + h^8 a_{0,8} + h^9 a_{0,9} \tag{2}$$

On the subinterval $[x_0, x_1]$, $\Delta_n: \ a = x_0 < x_1 < x_2 < \dots < x_n = b$, $x_i = a + ih$, where $i = 1, 2, .., n-1$ and $a_{0,j}$, $j = 5, 6, 8$ and 9 are unknowns to be determined. Let as examine subintervals $[x_i, x_{i+1}]$, $i = 1, 2, .., n-2$, by taking into account the interpolating conditions, form [12] provided that construction has been unique and the expression, for $p_i(x)$ will be in the follow form:

$$p_i(x) = y_i + ha_{i,1} + \frac{h^2}{2}y_i'' + h^3 a_{i,3} + \frac{h^4}{24}y_i^{(4)} + h^5 a_{i,5} + h^6 a_{i,6} + \frac{h^7}{5040}y_i^{(7)} + h^8 a_{i,8} + h^9 a_{i,9}, \tag{3}$$

where $a_{i,j}$, $i = 1, 2, ..., n-1$, $j = 1, 3, 5, 6, 8$ and 9, which are determined. Now we define the new approximate polynomial on the subinterval $[x_0, x_1]$, as

$$\bar{p}_0(x) = \bar{y}_0 + h\bar{y}_0' + \frac{h^2}{2}\bar{y}_0'' + \frac{h^3}{6}\bar{y}_0''' + \frac{h^4}{24}\bar{y}_0^{(4)} + h^5 \bar{a}_{0,5} + h^6 \bar{a}_{0,6} + \frac{h^7}{5040}\bar{y}_0^{(7)} + h^8 \bar{a}_{0,8} + h^9 \bar{a}_{0,9},$$

where $\bar{p}_0(x) = \bar{y}_1$; $\bar{p}_0''(x) = \bar{y}_0''$; $\bar{p}_0^{(4)}(x) = \bar{y}_1^{(4)}$ and $\bar{p}_0^{(7)}(x) = \bar{y}_1^{(7)}$

(4)

Form the above conditions in equation (4), and [12], we get the coefficients of $p_i(x)$ on $[x_0, x_1]$, as follows

$$\bar{a}_{0,5} = \frac{198}{41x^5}[\bar{y}_1 - \bar{y}_0] - \frac{198}{41x^4}\bar{y}_0' - \frac{239}{410x^2}\bar{y}_0''' - \frac{1}{410x^3}[91\bar{y}_1'' + 899\bar{y}_0''] + \frac{1}{984x}[5\bar{y}_1^{(4)} - 469\bar{y}_0^{(4)}] - \frac{x^2}{516600}[12\bar{y}_1^{(7)} - 35\bar{y}_0^{(7)}],$$

$$\bar{a}_{0,6} = -\frac{192}{41x^6}[\bar{y}_1 - \bar{y}_0] + \frac{192}{41x^5}\bar{y}_0' + \frac{617}{1230x^3}\bar{y}_0''' + \frac{1}{3099600x^4}[864360\bar{y}_1'' + 6393240\bar{y}_0''] - \frac{1}{3099600x^2}[23100\bar{y}_1^{(4)} - 195720\bar{y}_0^{(4)}] + \frac{x}{3099600}[122\bar{y}_1^{(7)} - 595\bar{y}_0^{(7)}],$$

$$\bar{a}_{0,8} = \frac{45}{41x^8}[\bar{y}_1 - \bar{y}_0] - \frac{45}{41x^7}\bar{y}_0' - \frac{9}{82x^5}\bar{y}_0'' - \frac{1}{826560x^6}[60480\bar{y}_1'' + 393129\bar{y}_0''] + \frac{1}{826560x^4}[2520\bar{y}_1^{(4)} - 10080\bar{y}_0^{(4)}] - \frac{1}{826560x}[23\bar{y}_1^{(7)} - 73\bar{y}_0^{(7)}],$$

and

$$\bar{a}_{0,9}=-\frac{10}{41x^9}[\bar{y}_1-\bar{y}_0]+\frac{10}{41x^8}\bar{y}_0'+\frac{1}{41x^6}\bar{y}_0'''+\frac{1}{2479680x^7}[40320\bar{y}_1''+262080\bar{y}_0'']-$$
$$\frac{1}{2479680x^5}[1680\bar{y}_1^{(4)}-6720\bar{y}_0^{(4)}]+\frac{1}{2479680x^2}[29\bar{y}_1^{(7)}+35\bar{y}_0^{(7)}].$$

The difference between $p_i(x)$ and $\bar{p}_i(x)$, yields the new polynomial denoted by $S_i(x)$ and defined on the interval $[x_0,x_1]$, as follows:

$$S_0(x)=p_0(x)-\bar{p}_0(x)$$
$$=x^5(a_{0,5}-\bar{a}_{0,5})+x^6(a_{0,6}-\bar{a}_{0,6})+x^8(a_{0,8}-\bar{a}_{0,8})+x^9(a_{0,9}-\bar{a}_{0,9}),$$
$$S_0'(x)=5x^4(a_{0,5}-\bar{a}_{0,5})+6x^5(a_{0,6}-\bar{a}_{0,6})+8x^7(a_{0,8}-\bar{a}_{0,8})+9x^8(a_{0,9}-\bar{a}_{0,9}),$$
$$S_0''(x)=20x^3(a_{0,5}-\bar{a}_{0,5})+30x^4(a_{0,6}-\bar{a}_{0,6})+56x^6(a_{0,8}-\bar{a}_{0,8})+72x^7(a_{0,9}-\bar{a}_{0,9}),$$
$$S_0'''(x)=60x^2(a_{0,5}-\bar{a}_{0,5})+120x^3(a_{0,6}-\bar{a}_{0,6})+336x^5(a_{0,8}-\bar{a}_{0,8})+504x^6(a_{0,9}-\bar{a}_{0,9}),$$
$$S_0^{(4)}(x)=120x(a_{0,5}-\bar{a}_{0,5})+360x^2(a_{0,6}-\bar{a}_{0,6})+1680x^4(a_{0,8}-\bar{a}_{0,8})+3024x^5(a_{0,9}-\bar{a}_{0,9}),$$
$$S_0^{(5)}(x)=120(a_{0,5}-\bar{a}_{0,5})+720x(a_{0,6}-\bar{a}_{0,6})+6720x^3(a_{0,8}-\bar{a}_{0,8})+15120x^4(a_{0,9}-\bar{a}_{0,9}),$$
$$S_0^{(6)}(x)=720(a_{0,6}-\bar{a}_{0,6})+20160x^2(a_{0,8}-\bar{a}_{0,8})+60480x^3(a_{0,9}-\bar{a}_{0,9}),$$
$$S_0^{(7)}(x)=40320x(a_{0,8}-\bar{a}_{0,8})+181440x^2(a_{0,9}-\bar{a}_{0,9}),$$
$$S_0^{(8)}(x)=40320(a_{0,8}-\bar{a}_{0,8})+362880x(a_{0,9}-\bar{a}_{0,9}),$$
$$S_0^{(9)}(x)=362880(a_{0,9}-\bar{a}_{0,9}).$$

Putting the value of $a_{0,j}$ and $\bar{a}_{0,j}$ where j=4,6,8 and 9 in $S_0(x)$ and $S_0^{(n)}(x)$, n=1,2,…,9, we obtains

$$S_0(x)=y_1-\bar{y}_1,$$
$$S_0'(x)=\frac{108}{41x}[y_1-\bar{y}_1]+\frac{51x}{410}[y_1''-\bar{y}_1'']-\frac{x^3}{984}[y_1^{(4)}-\bar{y}_1^{(4)}]+\frac{11x^6}{4132800}[y_1^{(7)}-\bar{y}_1^{(7)}],$$

(5)

$$S_0''(x) = y_1'' - \bar{y}_1'',$$

$$S_0'''(x) = -\frac{1080}{41x}[y_1 - \bar{y}_1] + \frac{154}{410x}[y_1'' - \bar{y}_1''] + \frac{23x}{984}[y_1^{(4)} - \bar{y}_1^{(4)}] - \frac{13x^4}{103320}[y_1^{(7)} - \bar{y}_1^{(7)}],$$

$$S_0^{(4)}(x) = y_1^{(4)} - \bar{y}_1^{(4)},$$

$$S_0^{(5)}(x) = \frac{36720}{41x^5}[y_1 - \bar{y}_1] - \frac{2940}{41x^3}[y_1'' - \bar{y}_1''] + \frac{225}{41x}[y_1^{(4)} - \bar{y}_1^{(4)}] + \frac{53x^2}{3444}[y_1^{(7)} - \bar{y}_1^{(7)}], \qquad (6)$$

$$S_0^{(6)}(x) = \frac{164160}{41x^6}[y_1 - \bar{y}_1] - \frac{11982}{41x^4}[y_1'' - \bar{y}_1''] + \frac{620}{41x^2}[y_1^{(4)} - \bar{y}_1^{(4)}] + \frac{752x}{4305}[y_1^{(7)} - \bar{y}_1^{(7)}],$$

III. Error Analysis:

In this section, we obtain some error bounds and convergence for $S_i(x)$ on $[x_i, x_{i+1}]$, $i = 1, 2,.., n-2$, as follows:

Theorem 1: Let $\bar{u}_k^{(r)}$ $(r = 0, 2, 4, 7; k = 0, 1, 2..., n)$ are the approximate values defined from equation (1). Then the following estimates of the spline function $\bar{S}_\Delta(x)$ are valid:

$\left|S_k^{(q)}(x)\right| = \left|P_k^{(q)}(x) - \bar{P}_k^{(q)}(x)\right| \le C_k\ \omega_9(h)$; for $q = 0, 1, ..., 9$; $k = 0, 1, ..., n-2$, where $x \in [x_i, x_{i+1}], i = 0, 1, 2, ..., n$ and C_k denote the constants dependent on h, and $\omega_9(h) = \omega(h, y^{(9)})$ is the modulus of continuity.

Proof: From [12], in the first interval $[x_0, x_1]$, we have

$$S_0(x) = p_0(x) - \bar{p}_0(x)$$
$$= x^5(a_{0,5} - \bar{a}_{0,5}) + x^6(a_{0,6} - \bar{a}_{0,6}) + x^8(a_{0,8} - \bar{a}_{0,8}) + x^9(a_{0,9} - \bar{a}_{0,9})$$

(4)

Hence

$$\left|S_0(x)\right| \le x^5\left|a_{0,5} - \bar{a}_{0,5}\right| + x^6\left|a_{0,6} - \bar{a}_{0,6}\right| + x^8\left|a_{0,8} - \bar{a}_{0,8}\right| + x^9\left|a_{0,9} - \bar{a}_{0,9}\right|$$
$$\le C_0\ \omega_9(h),$$

Similarly form equation (1), we have

$\left|S_0''(x)\right| \le C_2\ \omega_9(h)$, $\left|S_0^{(4)}(x)\right| \le C_4\omega_9(h)$ and $\left|S_0^{(7)}(x)\right| \le C_7\omega_9(h)$,

where C_0, C_2, C_5 and C_7 constants on depend of h. Also,

$$|s_0'(x)| = |P_0'(x) - \bar{P}_0'(x)| \le \frac{1}{4132800h}[10886400|u_1 - \bar{u}_1| + 514080h^2\,|u_1'' - \bar{u}_1''| -$$

$$4200h^4\,|u_1^{(4)} - \bar{u}_1^{(4)}| + 11h^7\,|u_1^{(7)} - \bar{u}_1^{(7)}|] \le \frac{1}{1100736h} C_1\ \omega_9(h),$$

where

$$C_1 = 10886400C_1^* + 514080h^2C_2^* - 4200h^4C_3^* + 11h^7C_4^*\ ,$$

$$|s_0'''(x)| \le \frac{1}{103320h^3}[-272160C_5^* + 38808h^2C_6^* + 2415h^4C_7^* - 13h^7C_8^*]$$

$$\le \frac{1}{103320\,h^3}\ C_3\ \omega_9(h)$$

where

$$C_3 = -272160\,C_5^* + 388080h^2C_6^* + 2415h^4C_7^* - 13h^7C_8^*,$$

$$|s_0^{(5)}(x)| \le \frac{1}{34444h^5} C_5\,\omega_9(h)\,,$$

where

$$C_5 = 3084480\,C_9^* - 246960h^2C_{10}^* + 18900h^4C_{11}^* + 53h^7C_{12}^*\,,$$

$$|s_0^{(6)}(x)| \le \frac{1}{4305h^6} C_6\,\omega_9(h)\ ,$$

Where $C_6 = 2(8618400\,C_{13}^* - 629055h^2C_{14}^* + 32550h^4C_{15}^* + 376h^7C_{16}^*)$,

$$|s_0^{(8)}(x)| \le \frac{1}{41h^8} C_8\,\omega_9(h)$$

Where $C_8 = 16(-113400\,C_{17}^* + 7560h^2C_{18}^* - 315h^4C_{19}^* + 8h^7C_{20}^*)$,

$$|s_0^{(9)}(x)| \le \frac{C_9}{41h^9}\,\omega_9(h)$$

Where $C_9 = 6(-604800C_{21}^* + 40320\,h^2C_{22}^* - 1680\,h^4C_{23}^* + 29\,h^7C_{24}^*)$, similarly on the interval $[x_i, x_{i+1}]$ can obtain the following:

$|s_i(x)| \le C_{10}\,\omega_9(h)$, $|s_i''(x)| \le C_{12}\ \omega_9(h)$, $|s_i^{(4)}(x)| \le C_{14}\ \omega_9(h)$, and

$|s_i^{(7)}(x)| \le C_{17}\,\omega_9(h)$.

and for the other derivatives can be find as follows

$|s_i'(x|) \le \frac{1}{4132800\,h} C_{11}\ \omega_9(h)$, where

$$C_{11} = 10886400\ C_{1i}^{*} - 6753600 * hC_{2i}^{*} + 514080\, h^{2} C_{3i}^{*} - 1572480\, h^{3} C_{4i}^{*} - 4200\, h^{5} C_{5i}^{*} + 11\, h^{7} C_{6i}^{*},$$

$$\left|s_i'''(x)\right| \le \frac{1}{103320h^{3}} C_{13}\omega_9(h), \left|s_i^{(5)}(x)\right| \le \frac{1}{3444h^{5}} C_{15}\, \omega_9(h)$$

$\left|s_i^{(6)}(x)\right| \le \frac{2}{4305h^{6}} C_{16}\, \omega_9(h), \left|s_i^{(8)}(x)\right| \le \frac{16}{41h^{8}} C_{18}\, \omega_9(h)$, and finally

$\left|s_i^{(9)}(x)\right| \le \frac{6}{41h^{9}} C_{19}\, \omega_9(h)$, where $C_1, C_3, C_5, C_6, C_8, C_{13}, C_{14}, C_{16}, C_{18}, C_{19}$ and C_i^{*} are constants depend on h.

Theorem 2: Consider $y(x)$ is the exact solution of any nonlinear initial value problems, and $\bar{P}_{\Delta}(x)$ be the approximate value of the ninth degree spline function approximation then $\left|y_k^{(q)}(x) - \bar{P}_k^{(q)}(x)\right| \le D_k\, \omega_9(h)$; for $q = 0, 1, ..., 9$, where $x \in [x_i, x_{i+1}]$, for $i = 1, 2, .., n-2$ and D_k^{*} denote the difference constants dependent on h, and $\omega_9(h) = \omega(h, y^{(9)})$.

Proof: see [3, 4, 12].

Theorem 3: If the function f in initial value problems satisfies all conditions in equation (4) and $\|.\|$ is a maximum norm, then the following inequalities are hold: $\left\|\bar{P}_{m-1}^{(r)}(x) - f[x, \bar{P}_0(x), \bar{P}_0'(x), .., \bar{P}_0^{(r)}(x)]\right\|_{\infty} \le G_{m-1,r}(h)\, \omega_9(h)$, where $G_{m-1,r}$ is constants dependent on h, $m = 1, 2, , ..., n-2$, $x \in [x_{m-1}, x_m]$ and $r = 0, 1, , ..., n-1$.

Proof: Using condition (3), we have

$\left\|D^{r}(f(x) - y(x))\right\|_{\infty} \le \frac{1}{1100736h} \omega_r(f; b-a), r = 1$ and $\left\|D^{r} y(x)\right\|_{\infty} \le \frac{1}{1100736h^{r}} \omega_r(f; 1)$

By using the Taylor series expanssion of y about zero, we get :

$$\left|D^{r}(y(x) - \bar{P}_{\Delta}(x))\right| \le \int_{0}^{u} \left|D^{r+1}(y - \bar{P}_{\Delta})(u)\right| du \le \left\|D^{r+1}(y - \bar{S}_{\Delta})\right\|_{\infty}$$

$$\le \left\|D^{r} y\right\|_{\infty} \le \frac{1}{C^{*} h^{r}} \omega_r(f; h).$$

After some derivations, we obtain:

$$\left\|D^{r}(\bar{S}_m(x) - f(x))\right\|_{\infty} \le \left\|D^{r}(\bar{S}_m - y)\right\|_{\infty} + \left\|D^{r} y - f\right\|_{\infty} \le G_{m-1,r}(h)\, \omega_9(f; h) ,$$

Where C^{*} and $G_{m-1,r}$ is constants dependent on h, $x \in [x_{m-1}, x_m]$ and $r = 0, 1, , ..., n-1$

Also, in a similar manner, the results on the other subintervals we proved.

IV. Numerical Results:

In this section, we apply the ninth degree spline for solving the third and second order nonlinear initial value problems. Numerical results for each problem are presented in tabular forms and compared with the approximate solution by ninth degree splines.

Problem (1): [9] consider the third order initial value problem

$$u''' + 2u'' + u' - u = \cos(t),$$

$$u(0) = 0, u'(0) = 1, u''(0) = 2.$$

Solution: let $t_0 = 0,\ u_0 = 0,\ u_0' = 1$ and $u_0'' = 2$, and using equations (5) and (6) with applying problem 1, we can find an approximate solution, and the errors estimation are shown on table 1.

Table 1: Errors estimation error for $S(x)$ and its derivative with different values of tolerance for the problem1:

h	$s_i(x)$	$s_i'(x)$	$s_i''(x)$	$s_i'''(x)$	$s_i^{(4)}(x)$
10^{-3}	0.1090	$1.6366*10^{-1}$ $1.6555*10^{-1}$ $1.6954*10^{-1}$	$1.9950*10^{0}$ $3.9940*10^{0}$ $6.9890*10^{0}$	$9.7909*10^{-1}$ $9.8292*10^{-1}$ $9.7646*10^{-1}$	$6.9890*10^{0}$ $9.9890*10^{0}$ $1.4976*10^{1}$
10^{-2}	$1.0903*10^{-4}$ $1.0098*10^{0}$ $1.9504*10^{0}$	$1.6585*10^{-1}$ $1.8512*10^{-1}$ $2.2337*10^{-1}$	$1.9504*10^{0}$ $3.9405*10^{0}$ $6.8908*10^{0}$	$9.6698*10^{-1}$ $1.0048*10^{0}$ $9.4112*10^{-1}$	$6.8908*10^{0}$ $9.8611*10^{0}$ $1.4757*10^{1}$

h	$s_i^{(5)}(x)$	$s_i^{(6)}(x)$	$s_i^{(7)}(x)$	$s_i^{(8)}(x)$	$s_i^{(9)}(x)$
10^{-3}	$3.8400*10^{-2}$ $5.4755*10^{-2}$ $8.2233*10^{-2}$	$3.9213*10^{-3}$ $9.5628*10^{-3}$ $2.1140*10^{-2}$	$2.3055*10^{1}$ $5.3881*10^{1}$ $1.1972*10^{2}$	$7.1976*10^{-2}$ $1.6821*10^{-1}$ $3.7376*10^{-1}$	$9.7843*10^{-5}$ $2.2867*10^{-4}$ $5.0808*10^{-4}$
10^{-2}	$3.8263*10^{-1}$ $5.3630*10^{-1}$ $8.1419*10^{-1}$	$3.0486*10^{-2}$ $1.0695*10^{-1}$ $2.2730*10^{-1}$	$2.3544*10^{1}$ $5.2824*10^{1}$ $1.1723*10^{2}$	$7.3504*10^{-1}$ $1.6491*10^{0}$ $3.6599*10^{0}$	$9.9920*10^{-3}$ $2.2418*10^{-2}$ $4.9752*10^{-2}$

Problem (2): Consider the second order nonlinear initial value problem

$$-4yy'' + 4(y')^2 - 0.5y^2 y' - 0.4y^2 y'y'' = 0$$
$$y(1) = 1,\ y'(1) = 0.5.$$

Solution: let $t_0 = 1,\ y_0 = 1$ and $y_0' = 0.5$, and using equations (5) and (6) with applying problem 2, we can find an approximate solution, and the errors estimation are shown on table 2.

Table 2: Errors estimation for $S(x)$ and its derivative with different values of tolerance for the problem2:

h	$s_i(x)$	$s_i'(x)$	$s_i''(x)$	$s_i'''(x)$	$s_i^{(4)}(x)$
10^{-3}	$7.5337*10^{-2}$	$2.5357*10^{-4}$	$5.0733*10^{-7}$	$8.0000*10^{-3}$	$2.5002*10^{-12}$
0.6	$7.5337*10^{-2}$	$2.5233*10^{-5}$	$5.0468*10^{-9}$	$8.0000*10^{-3}$	$2..4931*10^{-16}$

h	$s_i^{(5)}(x)$	$s_i^{(6)}(x)$	$s_i^{(7)}(x)$	$s_i^{(8)}(x)$	$s_i^{(9)}(x)$
10^{-3}	$3.0000*10^{-3}$	$2.5068*10^{-17}$	$1.4000*10^{-1}$	$3.0082*10^{-22}$	$6.7385*10^{-7}$
0.6	$3.4000*10^{-3}$	$2.4937*10^{-23}$	$1.4000*10^{-1}$	$2.9925*10^{-30}$	$6.7385*10^{-6}$

V. Conclusion:

In this paper, we use ninth degree spline function method for finding the solution of linear and nonlinear initial value problems. It may be concluded that ninth degree spline function is very powerful and efficient in finding the approximate solutions for a wide class of boundary value problems. The method gives more realistic series solutions that converge very rapidly in physical problems. The principal difference between an approximation spline interpolations and Lacunary interpolation from [12] showed slight superiority over the ninth degree spline function, the continuity of derivatives across elementedges improves convergence for all coefficients, and the approximate polynomial is established that reduces the errors estimation and convergence, also compared with that developed by [1, 2, 4] and [13].

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